International Journal of Theoretical Physics, Vol. 36, No. 4, 1997

Mechanical Systems with Nonlinear Constraints

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Received October 5, 1996

A geometrical formalism for nonlinear nonholonomic Lagrangian systems is developed. The solution of the constrained problem is discussed by using almost product structures along the constraint submanifold. Constrained systems with ideal constraints are also considered, and Chetaev conditions are given in geometrical terms. A Noether theorem is also proved.

1. INTRODUCTION

The theory of nonholonomic mechanical systems dates back to the last century. The constraint functions are functions on the position and velocities which constrain the motion to some submanifold of the phase space. Hertz (1894) gave a classification of constraints into holonomic (or integrable) and nonholonomic (or nonintegrable) constraints. If we are only in the presence of holonomic constraints, we reduce the configuration space to a submanifold on which the mechanical system is free. But if the constraints are purely nonholonomic, then not all the velocities are allowable, although all the positions are permitted.

The classification of constraints is very clear if we use a geometrical setting. The configuration space for a free system is a manifold Q (the configuration manifold) and the phase space of velocities is its tangent bundle TQ. The nonholonomic constraints are given by demanding that the only allowable velocities have to belong to a submanifold M of TQ. (The possible

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0020-7748/97/0400-0979$12.50/0

holonomic constraints were eliminated since they define Q.) The submanifold M is defined by the vanishing of a family of independent constraints $\Phi_i(q^A, v^A)$, $1 \leq i \leq m$, $A = 1, \ldots, \dim Q$, where (q^A, v^A) are fibered coordinates. The most usual case is when the submanifold M is linear, or, in other words, M is the total space of a vector subbundle of TQ. Thus, the constraints have the form $\Phi_i(q^A, v^A) = \Phi_{iA}(q^B)v^A$, where Φ_{iA} are functions on Q. In de Léon and Martín de Diego (1996b,e) we studied a geometrical setting for this kind of linear nonholonomic constraint. We only treated time-independent or scleronomic constraints, but in de Léon *et al.* (1996) we also studied the case of time-dependent or rheonomic constraints by using the geometrical setting of jet manifolds.

The purpose of this paper is to extend our results to the case of nonlinear constraints. In order to obtain the dynamics, it is necessary to assume some hypotheses on the constraints. This condition is the so-called admissibility of TM. Note that TM is a distribution on TQ along M, so we have extended the notion of admissibility introduced by Vershik and Faddeev (1972) (Definition 2.1). With this assumption, and if the system is regular (Definition 3.1), we obtain an almost product structure on TQ along M [that is, a pair of complementary projectors $(\mathcal{P}, \mathcal{Q})$] such that the projection $\mathcal{P}(\xi_L)$ of the Euler–Lagrange vector field ξ_L which gives the dynamics of the free Lagrangian L is just the solution of the constrained motion equations. It should be noted that if L is of natural type, i.e., $L = T - V$, where T is a kinetic energy coming from a Riemannian metric on Q, and V is a potential energy (a function on Q), the system is always regular.

If the constraints are ideal, i.e., the work of the forces of reaction of each constraint is equal to zero, the admissibility condition is nothing but the geometrical translation of the so-called Chetaev conditions. In this particular case, we construct a distribution H on TQ along M which is symplectic in (TTQ, ω_L), where ω_L is the symplectic Poincaré–Cartan two-form obtained from L. Thus, if we restrict ω_L and dE_L to H, where E_L is the energy function associated with L, the motion equation takes the usual form for a free Hamiltonian system at each fiber of H. This procedure extends that by Bates and Śniatycki (1992; Bates *et al.*, 1996) for the linear case. We define a new almost product structure on TQ along M associated with the decomposition $(TTQ)_{|M} = H \oplus H^{\perp}$, which gives, in fact, the dynamics by projecting ξ_L. We also prove a Noether theorem which generalizes one proved by Cushman *et al.* (1995) for linear constraints (see also de la Torre Juárez, 1996). Finally, we study some examples.

2. GEOMETRICAL THEORY OF CONSTRAINTS

Let Q be an n-dimensional manifold with tangent bundle TQ. The canonical projection will be denoted by τ_Q: $TQ \to Q$. Take bundle coordinates

(q^A, v^A). There exists a canonical (1, 1) tensor field J on TQ locally defined by

$$J = dq^A \otimes \frac{\partial}{\partial v^A}$$

J is called the canonical almost tangent structure of TQ (de Léon and Rodrigues, 1989), or sometimes, vertical endomorphism (Saunders, 1989). We denote by C the Liouville vector field on TQ; C is the infinitesimal generator of the dilations on TQ, and it is locally expressed by

$$C = v^A \frac{\partial}{\partial v^A}$$

We will extend the notion of admissible distribution on TQ introduced by Vershik and Faddeev (1972).

Definition 2.1. Let M be a submanifold of TQ and D a distribution on TQ along M. We say that D is admissible if

$$\dim D_x^o = \dim(J^*D^o)_x$$

for all $x \in M$, where D^o denotes the annihilator of D.

It is not hard to prove that D is admissible if and only if the linear mapping

$$J_x^*\colon\quad D_x^o \to (J^*D^o)_x$$

is an isomorphism. Thus, D is admissible if and only if D^o does not contain semibasic forms, since $\ker J_x^* = 0$.

From now on, we will assume that D is admissible.

Suppose now that $L\colon TQ \to \mathbb{R}$ is a regular Lagrangian function on TQ. Thus, the Hessian matrix $(\partial^2 L/\partial v^A \partial v^B)$ of L with respect to the velocities is nonsingular. If we define a two-form $\omega_L = -dJ^*(dL)$, we deduce that L is regular if and only if ω_L is symplectic. In this case, if $E_L = CL - L$ is the energy function associated with L, we know that the equation

$$i_X \omega_L = dE_L$$

has a unique solution ξ_L which is a second-order differential equation (SODE), say $J\xi_L = C$. Furthermore, the solutions of ξ are just the solutions of the Euler–Lagrange equations for L:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial v^A}\right) - \dfrac{\partial L}{\partial q^A} = 0 \\[2ex] \qquad\qquad v^A = \dfrac{dq^A}{dt} \end{cases} \tag{1}$$

ξ_L will be called the Euler–Lagrange vector field. From now on, we will assume that L is regular. Since ω_L is symplectic, it defines a Poisson bracket $\{\cdot,\cdot\}_L$ by

$$\{f, g\}_L = \omega_L(X_f, X_g)$$

for all functions f, g on TQ, where X_f is the Hamiltonian vector field given by $i_{X_f}\omega_L = df$. Thus, the evolution of an observable f is given by

$$\dot{f} = \xi_L(f) = \{f, E_L\}_L$$

If we denote by $\flat$: $\mathfrak{X}(TQ) \to \Lambda^1(TQ)$ the $C^\infty(TQ)$-module isomorphism given by $X \mapsto i_X\omega_L$, and $\# = \flat^{-1}$, we have $X_f = \#(df)$.

Consider the following system of equations:

$$\begin{cases} i_X\omega_L - dE_L \in J^*D^o \\ \qquad\qquad X \in D \cap TM \end{cases} \tag{2}$$

along M. The solutions of (2) will be termed the solutions of the problem (L, D, M) (on M). In most examples constraints arise as a submanifold of the whole tangent space TQ. In such a case, the solution of the constrained dynamics is a solution of the problem (L, D, M), where we now take $D = TM$. These considerations motivate our study of the problem (L, D, M).

Take a local basis $\{\mu_1, \ldots, \mu_m\}$ of D^o; hence $\{J^*\mu_1, \ldots, J^*\mu_m\}$ is a local basis of J^*D^o. Moreover, let $\{\Phi_\alpha, 1 \leq \alpha \leq 2n - \dim M\}$ be a family of independent functions on TQ defining M, or, in other words, a system of independent constraint functions. With these notations, equations (2) can be locally written as follows:

$$\begin{cases} i_X\omega_L = dE_L + \lambda^i J^*\mu_i \\ \mu_i(X) = 0 \\ d\Phi_\alpha(X) = 0 \end{cases} \tag{3}$$

along M, where λ^i are some Lagrange multipliers to be determined. It follows that any solution of (2) is a SODE along M.

We will denote by Z_i the symplectic gradient of the 1-form $J^*\mu_i$ with respect to ω_L, $Z_i = \#J^*\mu_i$. Notice that, because $J^*\mu_i$ is semibasic, Z_i is vertical. In fact, if $\mu_i = \mu_{iA}dq^A + \tilde{\mu}_{iA}dv^A$, we have

$$J^*\mu_i = \tilde{\mu}_{iA}dq^A$$

and hence $Z_i = -\tilde{\mu}_{iA}W^{AB}\,\partial/\partial v^B$, where (W^{AB}) is the inverse matrix of the Hessian matrix $(W_{AB} = \partial^2 L/\partial v^A\partial v^B)$.

By using Poisson brackets, we can equivalently write (3) as follows:

$$\begin{cases} \mu_k(\xi_L) + \lambda^i\mu_k(Z_i) = 0 \\ \{\Phi_\alpha, E_L\}_L + \lambda^i Z_i(\phi_\alpha) = 0 \end{cases} \tag{4}$$

along M.

The next result gives a necessary and sufficient condition to ensure the existence of solutions of equations (2). First, we introduce some notations. We set

$$S = \#(J^*D^o), \qquad S_L = \#(J^*D^o \oplus \langle dE_L \rangle) = S \oplus \langle \xi_L \rangle$$

Notice that we have assumed that $\xi_L(x) \notin S_x$ for $x \in M \cap \mathbb{O}$, where $\mathbb{O}$ is the zero section of TQ. In fact, this condition says that E_L is an independent constraint of the remainder. Thus, S and S_L are distributions on TQ along M.

Proposition 2.2. The problem (L, D, M) has a solution (on M) if and only if

$$S \cap (D \cap TM) \subsetneq S_L \cap (D \cap TM)$$

along M.

Proof. Assume that the problem (L, D, M) has a solution. This means that there exists at least a vector field X on M satisfying (2). Let $\{Z_i, 1 \leq i \leq m\}$ be a local basis of S, hence $\{Z_i, \xi_L\}$ is a local basis of S_L. Thus, $X = \xi_L + \lambda^i Z_i$ for some local functions λ^i. Because $X \in TD \cap S$, then $X \in S_L \cap (D \cap TM)$ and $X \notin S \cap (D \cap TM)$.

Conversely, assume that

$$S \cap (D \cap TM) \subsetneq S_L \cap (D \cap TM)$$

Thus there exists a vector field Y on M such that $Y \in S_L \cap (D \cap TM)$, but $Y \notin S \cap (D \cap TM)$. Therefore, $Y = \lambda \xi_L + \lambda^i Z_i$, where λ does not vanish at any point. This implies that $X = (1/\lambda)Y$ is a solution of (2). ■

Next, assume that the problem (L, D, M) has no solutions on M, or, equivalently, there exist points $x \in M$ such that $S_x \cap (D_x \cap T_xM) = (S_L)_x \cap (D_x \cap T_xM)$. We can develop a constraint algorithm as follows.

Put

$$M_2 = \{x \in M \mid S_x \cap (D_x \cap T_xM) \subsetneq (S_L)_x \cap (D_x \cap T_xM)\}$$

Thus, there exist solutions on M_2, but they are not necessarily tangent to M_2. Therefore, we put

$$M_3 = \{x \in M_2 \mid S_x \cap (D_x \cap T_xM_2) \subsetneq (S_L)_x \cap (D_x \cap T_xM_2)\}$$

and we obtain solutions which are tangent to M_2, but they are not necessarily tangent to M_3. We proceed further and obtain a sequence of constraint submanifolds

$$\cdots \to M_k \to \cdots M_3 \to M_2 \to M_1 = M$$

where for any $k > 1$ we have

$$M_{k+1} = \{x \in M_k | S_x \cap (D_x \cap T_xM_k) \subsetneq (S_L)_x \cap (D_x \cap T_xM_k)\}$$

Three possibilities may occur.

Case I. There exists an integer k such that $M_k = \emptyset$. In this case, the problem (L, D, M) has no solution.

Case II. There exists an integer k such that $M_k \neq \emptyset$, but $\dim M_k = 0$. In this case, M_k consists of isolated points and the only solution is $X = 0$.

Case III. There exists an integer k such that $M_{k+1} = M_k$, with $\dim M_k > 0$. Thus, there exists a vector field X on the final constraint submanifold $M_f = M_k$ such that

$$\begin{cases} i_X\omega_L - dE_L \in J^*D^o \\ \qquad X \in D \cap TM_f \end{cases} \tag{5}$$

along M_f.

Assume that $S \cap D = 0$. Therefore, we have $S \cap (D \cap TM) = 0$ and $0 \leq \dim(S_L)_x \cap (D_x \cap T_xM) \leq 1$ for all $x \in M$. If $\dim(S_L)_x \cap (D_x \cap T_xM) = 1$ for all $x \in M$, the system (L, D, M) has solution on M.

Proposition 2.3. If the Hessian matrix (W_{AB}) of L is positive or negative-definite, then $S \cap D = 0$.

Proof. Let $\{\mu_i = \mu_{iA}dq^A + \tilde{\mu}_{iA}dv^A\}$ be a local basis of D^o. Then $J^*\mu_i = \tilde{\mu}_{iA}dq^A$ and $\{Z_i = \#(J^*\mu_i) = -\tilde{\mu}_{iA}W^{AB}\,\partial/\partial v^B,\ i = 1, \ldots, n\}$ is a local basis of S. Since the matrix (W^{AB}) is positive or negative-definite, we deduce that the matrix $(\mu_j(Z_i)) = (-W^{AB}\tilde{\mu}_{iA}\tilde{\mu}_{jB})$ is regular. Therefore, we conclude that $S \cap D = 0$. ∎

3. MECHANICAL SYSTEMS WITH CONSTRAINTS

In this section we will apply the results of the preceding section to the particular case $D = TM$. This is the case of a mechanical system subjected to linear or nonlinear nonholonomic constraints.

Assume that D is just the distribution defined by the tangent bundle of the constraint submanifold M. Any function Φ: $TQ \to \mathbb{R}$ vanishing on M will be called a constraint. Since $D = TM$ is admissible, we have

$$\dim(T_xM)^o = \dim J^*((T_xM)^o)$$

for all $x \in M$.

Let $\{\Phi_i: TQ \to \mathbb{R},\ i = 1, \ldots, m\}$ be a set of independent functions defining the constraint submanifold M. In other words, $\{\Phi_i\}$ are a set of independent constraints. In principle, the functions $\Phi_i(q^A, v^A)$ are not necessarily linear in the velocities. A global basis for $(TM)^o$ is $\{d\Phi_i\}$, and equations (2) become

$$\begin{cases} i_X\omega_L - dE_L \in J^*(d\Phi_i) \\ \qquad\quad X \in TM \end{cases} \tag{6}$$

along M. The solutions of (6) will be termed the solutions of the problem (L, M). Since

$$J^*(d\Phi_i) = J^*\left(\frac{\partial\Phi_i}{\partial q^A}\, dq^A + \frac{\partial\Phi_i}{\partial v^A}\, dv^A\right) = \frac{\partial\Phi_i}{\partial v^A}\, dq^A$$

Equations (6) become

$$\begin{cases} i_X\omega_L = dE_L + \lambda^i \dfrac{\partial\Phi_i}{\partial v^A}\, dq^A \\ d\Phi_i(X) = 0 \end{cases} \tag{7}$$

along M.

Notice that if X_{Φ_i} is the Hamiltonian vector field associated with the constraint Φ_i, then $Z_i = -JX_{\Phi_i}$, Z_i being the vector field $Z_i = \#(J^*(d\Phi_i))$.

From Proposition 2.2 the system (L, M) has solution if and only if $S_x \cap T_xM \subsetneq (S_L)_x \cap T_xM$ for all $x \in M$. We introduce the following definition of regularity.

Definition 3.1. We say that the problem (L, M) is regular if $S \cap TM = 0$ along M.

In this case, we have

$$\dim S_x + \dim T_xM = \dim T_x(TQ) = 2n, \qquad \forall x \in M$$

and we obtain a decomposition of $T(TQ)$ as a Whitney sum

$$(TTQ)_{|M} = S \oplus TM$$

where $(TTQ)_{|M}$ denotes the restriction of the tangent bundle of TQ to the submanifold M. Hence, $\dim((S_L)_x \cap T_xM) = 1$ for all $x \in M$, and then there exists one and only one vector field $\xi_{L,M} \in S_L \cap TM$ which satisfies the SODE condition along M, that is, $J_x(\xi_{L,M})_x = C_x$ for all $x \in M$.

Remark 3.2. Observe that, from Proposition 2.3, if the Hessian matrix of L is positive or negative-definite, then the problem (L, M) is regular. For instance, if L is a natural Lagrangian (i.e., $L = T - V$, where $T = \frac{1}{2}g_{AB}v^A v^B$ is the kinetic energy of a Riemannian metric g on Q and V: $Q \rightarrow \mathbb{R}$ is a potential energy), then the problem (L, M) is regular (Bates and Śniatycki, 1992; Cariñena and Rañada, 1993; de Léon and Martín de Diego, 1996e).

Furthermore, there are two complementary projectors $\mathcal{P}$: $(TTQ)_{|M} \rightarrow TM$ and $\mathcal{Q}$: $(TTQ)_{|M} \rightarrow S$ associated with the above decomposition. We will obtain below an explicit expression in local coordinates.

Since the problem is regular, we know that the matrix $\mathcal{C} = (\mathcal{C}_{ij})$, where

$$\mathcal{C}_{ij} = d\Phi_i(Z_j) = Z_j(\Phi_i) = -W^{AB}\frac{\partial \Phi_i}{\partial v^A}\frac{\partial \Phi_j}{\partial v^B}$$

is regular, too. The projectors $\mathcal{P}$ and $\mathcal{Q}$ are respectively written as follows:

$$\mathcal{P} = \mathrm{Id} - \mathcal{C}^{ij}Z_i \oplus d\Phi_j, \qquad \mathcal{Q} = \mathcal{C}^{ij}Z_i \oplus d\Phi_j$$

Thus, the vector field $\xi_{L,M}$ can be obtained by projecting the Euler–Lagrange vector field ξ_L corresponding to the free Lagrangian system:

$$\xi_{L,M} = \mathcal{P}(\xi_L) = \xi_L - \mathcal{C}^{ij}\xi_L(\Phi_j)Z_i$$

So, the projection onto Q of the integral curves of $\xi_{L,M}$ satisfies the following Euler–Lagrange equations:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial v^A}\right) - \dfrac{\partial L}{\partial q^A} = -\lambda^i \dfrac{\partial \Phi_i}{\partial v^A} \\ \qquad\qquad v^A = \dfrac{dq^A}{dt} \end{cases}$$

Remark 3.3. It should be noticed that $(\mathcal{P}, \mathcal{Q})$ defines an almost product structure on the manifold TQ along the submanifold M.

Consider now the vector bundle isomorphism $\tilde{\mathcal{P}}$: $T^*TQ \rightarrow T^*TQ$ defined along M by

$$\tilde{\mathcal{P}} = \flat \circ \mathcal{P} \circ \#$$

Proposition 3.4. The vector field $\xi_{L,M}$ is the unique solution of the equation

$$i_X \omega_L = \tilde{\mathcal{P}}(dE_L) \tag{8}$$

along M.

Proposition 3.4 is an alternative version of the above procedure to obtain the dynamics; we project the energy instead of the free Euler–Lagrange vector field.

3.1. Ideal Constraints

In what follows, we continue under the assumption $D = TM$.

Definition 3.5. A constraint function Φ is said to be ideal if the work of the forces of reaction of the constraint is equal to zero. The system (L, M) is said to be ideal if all the constraints are ideal.

A constraint Φ is ideal if and only if $J^*(d\Phi)(X) = 0$ for a solution X of (6). Since X satisfies the SODE condition, we obtain that a constraint Φ is ideal if and only if

$$v^A \frac{\partial \Phi}{\partial v^A} \approx 0$$

which is equivalent to $C(\Phi) \approx 0$ (the symbol $\approx$ means weakly equal or, in other words, equality on M). For instance, if a constraint is homogeneous of degree r on the velocities, then it is ideal because

$$v^A \frac{\partial \Phi}{\partial v^A} = r\Phi \approx 0$$

In particular, a linear constraint is always ideal.

Thus, we get the following geometrical interpretation of ideal constraints.

Proposition 3.6. The system (L, M) is ideal if and only if the Liouville vector field C is tangent to M.

Remark 3.7. It should be noticed that the condition of admissibility for TM is nothing but the usual Chetaev condition for the constraints (Pironneau, 1982).

Denote by D^v the distribution along M whose annihilator is $J^*(TM)^o$.

Lemma 3.8. If ξ is a SODE, then for any point $x \in M$, we have

$$\xi(x) \in (D^v)_x \Leftrightarrow C(x) \in T_xM$$

Proof. It follows from a direct computation. ∎

Consider the inclusion $j: M \to TQ$ and the complementary projectors $\mathcal{P}$ and $\mathcal{Q}$ defined in the above section. We obtain the following.

Proposition 3.9. If ω_M is the restriction of ω_L to the constrained submanifold M, then the solution $\xi_{L,M}$ of the constrained dynamics satisfies the equation

$$i_X\omega_M = j^*(\tilde{\mathcal{P}}(dE_L)) \tag{9}$$

Moreover, the unique vector field on M satisfying the SODE condition and equation (9) is just $\xi_{L,M}$.

Proof. Since the vector field $\xi_{L,M}$ satisfies (8), then it also satisfies (9).

Now, let X be a vector field on M satisfying the SODE condition [that is, $(JX = C)_{|M}$] such that $i_X\omega_M = j^*(\tilde{\mathcal{P}}(dE_L))$. Then, we have that

$$(i_X\omega_L)(\mathcal{P}(Y)) = \tilde{\mathcal{P}}(dE_L)(\mathcal{P}(Y))$$

for all vector fields Y on TQ along M. Moreover, from Lemma 3.8 and using that $\mathcal{Q}Y \in S$ and the fact that X satisfies the SODE condition, we obtain that $(i_X\omega_L)(\mathcal{Q}Y) = \tilde{\mathcal{P}}(dE_L)(\mathcal{Q}Y) = 0$.

Therefore, we conclude that $i_X\omega_L = \tilde{\mathcal{P}}(dE_L)$, which implies that $X = \xi_{L,M}$. ■

Remark 3.10. Notice that ω_M is no longer symplectic, so that it may be another solution of the equation

$$i_X\omega_M = j^*(\tilde{\mathcal{P}}(dE_L))$$

Next, we will prove the following two lemmas.

Lemma 3.11. $(D^v)_x$ is coisotropic in $(T_x(TQ), \omega_L(x))$ for all $x \in M$.

Proof. In fact, since $J^*(TM)^o$ is locally generated by semibasic 1-forms, we deduce that

$$(D^v)_x^\perp = S_x \subset V_x\,(TQ) \subset (D^v)_x$$

for all $x \in M$, where $(D^v)_x^\perp$ denotes the $\omega_L(x)$ complement of $(D^v)_x$. ■

Lemma 3.12. The problem (L, M) is regular (that is, $S \cap TM = 0$) if and only if the distribution $H = D^v \cap TM$ along M is symplectic in (TTQ, ω_L).

Proof. If $S \cap TM = 0$, then

$$S \cap TM = TM \cap (D^v)^\perp = 0$$

and

$$T_x(TQ) = (D^v)_x^\perp \oplus T_xM, \qquad \forall x \in M$$

Hence, from Lemma 3.11 we obtain

$$(D^v)_x = (D^v)_x^\perp \oplus (T_xM \cap (D^v)_x) = (D^v)_x^\perp \oplus H_x$$

Therefore, we deduce that H_x is a symplectic subspace in $(T_x(TQ), \omega_L(x))$ for all $x \in M$. Conversely, assume that H is symplectic in (TTQ, ω_L). Take

$$Z \in S_x \cap T_xM = (D^v)_x^\perp \cap T_xM \subset (D^v)_x \cap T_xM = H_x$$

Because $\omega_L(x)(Z, W) = 0$ for all $W \in H_x$, we conclude that $Z = 0$. ■

Consider now, as in Bates and Śniatycki (1992) and Cushman *et al.* (1995), the restrictions ω_H and $d_H E$ to H of the symplectic form ω_L and dE_L, respectively. Since H is symplectic, there exists a unique solution on H of the equation

$$i_X \omega_H = d_H E \tag{10}$$

Proposition 3.13. The solution of equation (10) is the vector field $\xi_{L,M}$.

Proof. Since $\xi_{L,M}$ is a solution of the problem (L, M), then, for each point $x \in M$, $\xi_{L,M}(x) \in T_x M$. From Lemma 3.8, $\xi_{L,M} \in D^v$. Therefore, $\xi_{L,M} \in H$ and it is trivially a solution of equation (10). ∎

Since H is symplectic we obtain a new decomposition of the whole tangent space (of course, along the points of M):

$$(TTQ)_{|M} = H \oplus H^\perp$$

and two new complementary projectors (associated with the above decomposition) arise: $\overline{\mathcal{P}}$: $(TTQ)_{|M} \to H$ and $\overline{\mathfrak{Q}}$: $(TTQ)_{|M} \to H^\perp$.

If $\{\Phi_i, i = 1, \ldots, m\}$ is a set of independent functions defining the constraint submanifold M, then $H^\perp$ is locally generated by the vector fields X_{Φ_i} and Z_i $[Z_i = \#(J^*(d\Phi_i))]$, $1 \leq i \leq m$. In such a case, the projector $\overline{\mathfrak{Q}}$ is locally defined as follows:

$$\overline{\mathfrak{Q}} = \mathscr{C}^{ij} Z_j \otimes d\Phi_i - \mathscr{C}^{ij} X_{\Phi_j} \otimes J^* d\Phi_i + \mathscr{C}^{jk} \mathscr{C}^{il} \{\Phi_k, \Phi_l\}_L Z_j \otimes J^*(d\Phi_i)$$

where $\mathscr{C}^{ij}$ are the entries of the inverse matrix of $\mathscr{C} = (\mathscr{C}_{ij}) = (Z_i(\Phi_j))$.

Moreover, the projection by $\overline{\mathcal{P}}$ of the Euler–Lagrange vector field also gives the solution of the constrained dynamics $\xi_{L,M}$, that is,

$$\overline{\mathcal{P}}(\xi_L) = \xi_{L,M}$$

Next, we will prove a Noether theorem which generalizes that proved in Cushman *et al.* (1995).

A function f: $TQ \to \mathbb{R}$ is said to be a constant of the motion if $\xi_{L,M}(f_{|M}) = 0$.

Theorem 3.14 (Noether theorem). A function f: $TQ \to \mathbb{R}$ is a constant of the motion of $\xi_{L,M}$ if and only if the energy is constant along the integral curves of the vector field $\overline{\mathcal{P}}(X_f)$, that is, $(\overline{\mathcal{P}}(X_f))(E_L) = 0$.

Proof. Since

$$i_{\xi_{L,M}} \omega_L - dE_L \in J^*(TM)^o$$

then $(i_{\xi_{L,M}} \omega_L - dE_L)(\overline{\mathcal{P}}(X_f)) = 0$. Therefore,

$$\omega_L(\xi_{L,M}, \overline{\mathcal{P}}(X_f)) = \overline{\mathcal{P}}(X_f)(E_L)$$

Also, because $\xi_{L,M} \in H$, we deduce that

$$\omega_L(\xi_{L,M}, \overline{\mathcal{P}}(X_f)) = \omega_L(\xi_{L,M}, X_f)$$

Thus,

$$-\xi_{L,M}(f) = (\overline{\mathcal{P}}(X_f))(E_L)$$

which proves our result. ■

Corollary 3.15. The energy function E_L and the constraint functions $\Phi_1, \ldots, \Phi_m$ are constants of the motion.

Proof. In fact,

$$\overline{\mathcal{P}}(X_{E_L}) = \overline{\mathcal{P}}(\xi_L) = \xi_{L,M}$$

and

$$\overline{\mathcal{P}}(X_{\Phi_i}) = 0 \quad ■$$

Remark 3.16. Obviously, the restriction of Φ_i to M vanishes, and then we do not obtain information concerning the integration of the dynamical system $\xi_{L,M}$ on M. However, E_L restricted to M is a first integral of $\xi_{L,M}$.

Remark 3.17. As in Section 2, if the problem (L, M) is singular, that is, $S \cap TM \neq 0$, or, equivalently, $H = D^v \cap TM$ is not symplectic in (TTQ, ω_L), we have a constraint algorithm which permits us to obtain a final constraint submanifold where a well-defined solution of the dynamics exists.

This algorithm gives a sequence of constraint submanifolds

$$\cdots \to M_k \to \cdots M_3 \to M_2 \to M_1 = M$$

where for any $k > 1$ we have

$$M_{k+1} = \{x \in M_k \mid S_x \cap T_xM_k \subsetneq (S_L)_x \cap T_xM_k\}$$

Recall that $S_L = S \oplus \langle \xi_L \rangle$. If the above sequence stabilizes, i.e., there exists an integer k such that $M_k = M_{k+1}$, then there is at least a vector field X on $M_f = M_k$ such that

$$\begin{cases} i_X\omega_L - dE_L \in J^*(TM^o) \\ \qquad X \in TM_f \end{cases} \tag{11}$$

along M_f.

4. EXAMPLES

Example 4.1 (Linear Constraints) (de Léon and Martín de Diego, 1996a,c–e; see also Giachetta, 1992; Koiller, 1992; Massa and Pagani, 1991, 1995; Neimark and Fufaev, 1972; Rosenberg, 1977; Rumiantsev, 1978; Sarlet, 1996a,b; Sarlet *et al.*, 1995; Saunders *et al.*, 1996; Vershik and Gershkovich, 1994). Assume that M is a linear submanifold of TQ, that is, M is the total space of a vector subbundle of TQ or, in other words, M is a distribution on the configuration manifold Q. We denote by M^v and M^c the distributions on TQ whose annihilators are defined as follows. If $M^o = \langle \mu_i \rangle$, then

$$(M^v)^o = \langle \mu_i^V \rangle, \qquad (M^c)^o = \langle \mu_i^V, \mu_i^C \rangle$$

where α^V and α^C denote the vertical and complete lifts of a 1-form α on Q to TQ, respectively. The constrained motion equations can be written as follows:

$$\begin{cases} (i_X\omega_L - dE_L) \in (M^v)^o \\ \qquad\qquad X \in M^c \end{cases} \tag{12}$$

along M.

Moreover, we can consider M as a submanifold of TQ. A straightforward computation shows that equations (12) are equivalent to equations (2) for the problem (L, M), namely

$$\begin{cases} i_X\omega_L - dE_L \in J^*(TM)^o \\ \qquad\qquad X \in TM \end{cases}$$

along M.

Example 4.2 (Pironneau, 1982). Consider a little ring which rotates, without slipping, about a horizontal fixed curve.

The Lagrangian function L: $T\mathbb{R}^3 \to \mathbb{R}$ is given by

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

and we take the quadratic constraint

$$\Phi = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - c \qquad (c > 0)$$

which means that the module of the velocity would be constant. Notice that Φ is not an ideal constraint.

After some computations we obtain

$$E_L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$$

$$\omega_L = m\, dx \wedge d\dot{x} + m\, dy \wedge d\dot{y} + m\, dz \wedge d\dot{z}$$

$$\xi_L = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} - g\frac{\partial}{\partial \dot{z}}$$

Let M be the submanifold of $T\mathbb{R}^3$ determined by the vanishing of the function Φ. Notice that M is a fiber bundle over $\mathbb{R}^3$ with typical fiber the two-sphere S^2.

The annihilator of TM is

$$(TM)^o = \langle d\phi \rangle = \langle \dot{x}\, d\dot{x} + \dot{y}\, d\dot{y} + \dot{z}\, d\dot{z} \rangle$$

Therefore,

$$J^*(TM)^o = \langle \dot{x}\, dx + \dot{y}\, dy + \dot{z}\, dz \rangle$$

and S is generated by the vector field

$$Z = -\frac{2}{m}\left(\dot{x}\frac{\partial}{\partial \dot{x}} + \dot{y}\frac{\partial}{\partial \dot{y}} + \dot{z}\frac{\partial}{\partial \dot{z}}\right)$$

According to Remark 3.2, the system is regular.

The solution of the constrained problem is

$$\xi_{L,M} = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} + \frac{g\dot{z}\dot{x}}{c}\frac{\partial}{\partial \dot{x}} + \frac{g\dot{z}\dot{y}}{c}\frac{\partial}{\partial \dot{y}} + \left(\frac{g\dot{z}^2}{c} - g\right)\frac{\partial}{\partial \dot{z}}$$

Notice that the restriction of E_L to M

$$(E_L)_{|M} = \tfrac{1}{2}mc + mgz$$

is not a first integral of $\xi_{L,M}$.

Example 4.3 (Appel's Example) (Pironneau, 1982). The mechanical system is described by the Lagrangian function $L: T\mathbb{R}^3 \to \mathbb{R}$ given by

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

and subjected to a quadratic constraint

$$\Phi = a^2(\dot{x}^2 + \dot{y}^2) - \dot{z}^2$$

Notice that Φ is an ideal constraint.

E_L, ω_L, and ξ_L are the same as in the above example.

Let M be the submanifold of $T\mathbb{R}^3$ defined by

$$M = \{(x, y, z, \dot{x}, \dot{y}, \dot{z}) \in T\mathbb{R}^3 \mid \Phi(x, y, z, \dot{x}, \dot{y}, \dot{z}) = 0 \text{ and } \dot{z} \neq 0\}$$

The annihilator of TM is

$$(TM)^o = \langle d\phi \rangle = \langle a^2\dot{x}\, d\dot{x} + a^2\dot{y}\, d\dot{y} - \dot{z}\, d\dot{z} \rangle$$

Therefore, we get

$$J^*(TM)^o = \langle a^2\dot{x}\, dx + a^2\dot{y}\, dy - \dot{z}\, dz \rangle$$

and S is generated by the vector field

$$Z = a^2\dot{x}\frac{\partial}{\partial \dot{x}} + a^2\dot{y}\frac{\partial}{\partial \dot{y}} - \dot{z}\frac{\partial}{\partial \dot{z}}$$

According to Remark 3.2, the system is regular.

The solution of the constrained problem is

$$\xi_{L,M} = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} - \frac{ga^2}{\dot{z}(1+a^2)}\dot{x}\frac{\partial}{\partial \dot{x}} - \frac{ga^2}{\dot{z}(1+a^2)}\dot{y}\frac{\partial}{\partial \dot{y}} - \frac{ga^2}{1+a^2}\frac{\partial}{\partial \dot{z}}$$

Therefore, the motion equations are

$$\left\{\begin{aligned} \ddot{x} &= -\frac{ga^2}{1+a^2}\frac{\dot{x}}{\dot{z}} \\ \ddot{y} &= -\frac{ga^2}{1+a^2}\frac{\dot{y}}{\dot{z}} \\ \ddot{z} &= -\frac{ga^2}{1+a^2} \end{aligned}\right. \tag{13}$$

From equations (13) we deduce that

$$\frac{\ddot{x}}{\dot{x}} = \frac{\ddot{y}}{\dot{y}} = \frac{\ddot{z}}{\dot{z}}$$

and then the velocities are constrained to follow a fixed direction.

The function $f = \dot{z}^2 + [2a^2g/(1 + a^2)]z$ is a constant of the motion of $\xi_{L,M}$ (in fact, f is a combination of E_L and Φ).

ACKNOWLEDGMENTS

This work has been partially supported through grants DGICYT (Spain), Project PB94-0106, Consejería de Educación del Gobierno de Canarias, and UNED (Spain).

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